

A Simple Condition for the Existence of Transversals

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Abstract

Hall's Theorem is a basic result in Combinatorics which states that the obvious necessary condition for a finite family of sets to have a transversal is also sufficient. We present a sufficient (but not necessary) condition on the sizes of the sets in the family and the sizes of their intersections so that a transversal exists. Using this, we prove that in a bipartite graph G (bipartition $\{A, B\}$), without 4-cycles, if $\deg(v) \geq \sqrt{2e|A|}$ for all $v \in A$, then G has a matching of size $|A|$.

1 Introduction and Preliminaries

In this paper, we look at a sufficient (but not necessary) condition for a finite family of sets to have a transversal, which only involves the sizes of the sets and the sizes of their intersections.

We begin by recalling some theorems and definitions necessary for our presentation.

Definition 1 (Transversal). Let $\mathcal{F} = \{S_1, \dots, S_n\}$ be a finite family of sets. A transversal for \mathcal{F} is a tuple (T, ϕ) , where $T \subseteq \bigcup \mathcal{F}$, and $\phi : T \rightarrow \mathcal{F}$ is a bijection such that $x \in \phi(x)$ for all $x \in T$.

Theorem 1 (Hall's Theorem). Let $\mathcal{F} = \{S_1, \dots, S_n\}$ be a finite family of sets. If for each $\mathcal{F}' \subseteq \mathcal{F}$,

$$\left| \bigcup \mathcal{F}' \right| \geq |\mathcal{F}'|,$$

then \mathcal{F} has a transversal.

For an elegant proof of the above result, the reader may refer to [1].

Consider a finite set of objects S and a property P . Suppose we have a probability distribution on S . If an element $x \in S$ picked randomly according to the distribution has property P with positive probability, then the set S has atleast one element with property P . This is the basic idea behind the "Probabilistic Method", which has been used to produce elegant proofs of many combinatorial results.

In what follows, we formalise this idea.

Definition 2 (Dependency Digraph). Let E_1, \dots, E_n be events in a probability space. The *dependency digraph* for these events is the graph $G = (V, E)$, where $V = \{1, \dots, n\}$ and $E = \{(i, j) \mid 1 \leq i, j \leq n, i \neq j, E_i \text{ and } E_j \text{ are dependent events}\}$.

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Lemma 1 (The Lovász Local Lemma). *Let E_1, \dots, E_n be events in a probability space and D be their dependency digraph. Suppose there are numbers $0 \leq x_i < 1$, $1 \leq i \leq n$ such that*

$$P(E_i) \leq x_i \prod_{(i,j) \in E(D)} (1 - x_j), \text{ for all } 1 \leq i \leq n.$$

Then

$$P\left(\bigwedge_{i=1}^n E_i^c\right) \geq \prod_{i=1}^n (1 - x_i).$$

In particular, the probability that none of the events E_i occur is positive.

Corollary 1. *Let E_1, \dots, E_n be events in a probability space such that each event is dependent on at most d other events and $P(E_i) \leq p$, for all $1 \leq i \leq n$. If*

$$ep(d+1) \leq 1,$$

then

$$P\left(\bigwedge_{i=1}^n E_i^c\right) > 0.$$

Proofs of the above lemma and its corollary can be found in [2].

Now suppose we want to prove that in a set of objects, there is an object with a certain property. If we can identify “bad events”, which prevent a randomly picked object from having the property, then by the above corollary, it suffices to show that each “bad event”, occurs with small enough probability and it does not depend on too many other “bad events”.

In the next section, using the corollary, we obtain a simple sufficient condition for the existence of a transversal for a finite family of *finite* sets.

2 The Condition

We are now ready to present our result.

Theorem 2. *Let $\mathcal{F} = \{S_1, \dots, S_n\}$ be a finite family of finite sets. Suppose there are numbers l and m such that $|S_i| \geq l$ and $|S_i \cap S_j| \leq m$, for all $1 \leq i, j \leq n$. If*

$$\sqrt{em(2n-3)} \leq l,$$

then \mathcal{F} has a transversal.

Proof. Consider a random n -tuple (X_1, \dots, X_n) , where X_i is a uniform random variable over S_i , for $1 \leq i \leq n$. For $1 \leq i, j \leq n$, denote by E_{ij} the event $\{X_i = X_j\}$. The random tuple is a transversal when none of the events E_{ij} , $1 \leq i < j \leq n$ occur. This is exactly the event $\bigwedge_{1 \leq i < j \leq n} E_{ij}^c$.

We have

$$\begin{aligned} P(E_{ij}) &= \frac{|S_i \cap S_j|}{|S_i||S_j|} \\ &\leq \frac{m}{l^2}. \end{aligned}$$

Each event E_{ij} only depends on another event $E_{i'j'}$ if either $i = i'$ or $j = j'$. This can happen in $(n-2) + (n-2) = 2n-4$ ways. Applying the Local lemma with $p = \frac{m}{l^2}$ and $d = 2n-4$, we have

$$P\left(\bigwedge_{1 \leq i < j \leq n} E_{ij}^c\right) > 0,$$

if

$$\begin{aligned} ep(d+1) &= e \frac{m}{l^2} (2n-4+1) \leq 1, \text{ i.e.} \\ em(2n-3) &\leq l^2, \text{ i.e.} \\ \sqrt{em(2n-3)} &\leq l. \end{aligned}$$

Thus the event $\bigwedge_{1 \leq i < j \leq n} E_{ij}^c$ occurs with positive probability, i.e. the family \mathcal{F} has a transversal when $\sqrt{m(2n-3)} \leq l$. \square

3 Matchings in Graphs

By considering the set of neighbours of each vertex of a graph as an element of a family of sets, we can use the previous theorem to obtain the following result about the existence of matchings which saturate one of the vertex sets in a bipartition of a graph.

Theorem 3. *Let $G = (V, E)$ be a bipartite graph with no 4-cycles. Let $\{A, B\}$ be a bipartition of V and $n = |A|$. If $\deg(v) \geq \sqrt{2en}$ for all $v \in A$, then G has a matching which saturates A .*

Proof. For each $v \in A$, define

$$S_v = \{u \in B \mid u \text{ is a neighbour of } v\}.$$

Since $\{A, B\}$ is a bipartition, all neighbours of vertices in A are in B . Thus we have

$$|S_v| = \deg(v) \geq \sqrt{2en}.$$

Claim 1 (For all $u, v \in A$, $|S_u \cap S_v| \leq 1$). Suppose $|S_u \cap S_v| \geq 2$. Then there are vertices $u', v' \in B$ such that uu', uv', vu', vv' are edges in G . Thus G contains the 4-cycle $uu'vv'u$. This contradicts our assumption that G is 4-cycle-free.

Now we apply our theorem from the previous section with $l = \sqrt{2en}$ and $m = 1$. Since

$$\sqrt{em(2n-3)} = \sqrt{e(2n-3)} \leq \sqrt{2en} = l,$$

the family $\mathcal{F} = \{S_v \mid v \in A\}$ has a transversal. Let (B', ϕ) be the transversal. Then

$$M = \{\{v, \phi^{-1}(S_v)\} \mid v \in A\}$$

is a matching which saturates A . \square

References

- [1] Martin Aigner and Günter M. Ziegler. *Proofs from the Book*. Springer, 2010.
- [2] Noga Alon and Joel H. Spencer. *The Probabilistic Method*. John Wiley and Sons, 2008.